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# ON PARAMETRIZATION, OPTIMIZATION AND TRIVIALITY OF TEST CONFIGURATIONS

YUJI ODAKA

ABSTRACT. We give a parametrization of test configurations in the sense of Donaldson via spherical buildings, and show the existence of “optimal” destabilizing test configurations for unstable varieties, in the wake of Mumford and Kempf. We also give an account of the recent slight amendment to the definition of K-stability after Li-Xu, from two other viewpoints: from the one parameter subgroups and from the author’s blow up formalism.

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## 1. INTRODUCTION

The concept of test configurations of polarized varieties is introduced by Donaldson [Don02] to generalize K-stability after Tian [Tia97] which is a certain version of GIT stability [Mum65] for polarized varieties. Indeed, to each test configuration, certain GIT weights correspond, which is a motivation of its introduction. Precisely speaking, they can be regarded as certain *equivalent classes* of one parameter subgroups of the general linear group  $\mathrm{GL}(H^0(X, L^{\otimes r}))$  for a given polarized variety  $(X, L)$ .

The aim of this short note is twofold. First is to point out that, roughly speaking, dividing sets of test configurations further, all equivalent classes can be parametrized by a certain simplicial complex, the so-called *spherical building* invented by J. Tits (cf. [Tit74]). Moreover, thanks to a theorem of Rousseau [Rou78] and Kempf [Kem78], originally known as the Tit’s center conjecture, we conclude the existence of “optimal” destabilizing test configurations. The outcome is a geometric analogue of the Harder-Narasimhan filtrations of unstable sheaves on a fixed variety. (The analogy becomes clearer after the recent work of [GSZ11]. Namely, it claims that the filtration coincides the optimal destabilizing objects in the sense of [Rou78], [Kem78]. ) We remark here that the spherical building has already been

discussed in the Mumford's classic [Mum65] to give applications to stability problem although he used the word “flag complex” with completely the same meaning. Thus, we should note that our statements are essentially consequences of [Mum65], [Rou78], [Kem78], and only some small pieces of arguments, so that they are not essentially new. However, it seems to the author that the results have not been recognized by the mathematical community of this area, which is one motivation for him of writing this notes.

The second objective of this notes is to explain from our point of view, about the pathological (almost trivial but not trivial) test configurations whose existence is pointed out in [LX11, subsection 2.2]. They addressed that we need to slightly modify the definition of K-stability by regarding them as trivial or “throwing them away”. For details, see subsection 3.3. In that subsection, we also explain how the blow up formalism [Od11] works for that amended notion of K-stability. Moreover, the sub-locus of the spherical building which parametrizes those test configurations of specific kind will be described as well.

## 2. STATEMENTS

We work over an arbitrary algebraically closed field  $k$ . We use the terminologies of line bundles and invertible sheaves interchangeably. By a polarized variety  $(X, L)$ , we mean that  $X$  is a projective equidimensional variety and  $L$  is an ample line bundle on it. At the beginning, we recall the definition of (semi) test configurations and introduce certain (equivalence) relations for test configurations as follows.

**Definition 2.1.** A *test configuration* (resp. *semi test configuration*) for a polarized variety  $(X, L)$  is a quasi-projective variety  $\mathcal{X}$  with an invertible sheaf  $\mathcal{L}$  on it with:

- (i) a  $\mathbb{G}_m$  action on  $(\mathcal{X}, \mathcal{L})$
- (ii) a projective flat morphism  $p: \mathcal{X} \rightarrow \mathbb{A}^1$

such that  $p$  is  $\mathbb{G}_m$ -equivariant for the usual action on  $\mathbb{A}^1$ :

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ (t, x) & \longmapsto & tx, \end{array}$$

$\mathcal{L}$  is relatively ample (resp. relatively semi ample), and  $(\mathcal{X}, \mathcal{L})|_{p^{-1}(\mathbb{A}^1 \setminus \{0\})}$  is  $\mathbb{G}_m$ -equivariantly isomorphic to  $(X, L^{\otimes r}) \times (\mathbb{A}^1 \setminus \{0\})$  for some positive integer  $r$ , called *exponent*, with the natural action of  $\mathbb{G}_m$  on the latter and the trivial action on the former.

**Definition 2.2.** For a given polarized variety  $(X, L)$ , we introduce the following three pre-equivalent conditions. We call two test configurations  $(\mathcal{X}_i, \mathcal{L}_i) (i = 1, 2)$

- *P-pre-equivalent* if  $(\mathcal{X}_2, \mathcal{L}_2)$  is a base change of  $(\mathcal{X}_1, \mathcal{L}_1)$  by the power morphism  $t \mapsto t^m$  of  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$  with some  $\mathbb{Z}_{>0}$ .
- *Q-pre-equivalent* if  $(\mathcal{X}_2, \mathcal{L}_2) = (\mathcal{X}_1, \mathcal{L}_1^{\otimes l})$  equivariantly with  $l \in \mathbb{Z}_{>0}$ .
- *R-equivalent* when there is a  $\mathbb{G}_m$ -equivariant isomorphism  $\phi: \mathcal{X}_1 \cong \mathcal{X}_2$  over  $\mathbb{A}^1$  with  $\mathcal{L}_1 \cong \phi^* \mathcal{L}_2$ . To say in other (but a little rough) words, it is exactly when they are isomorphic “except for the linearizations”.

Consider an equivalent relation among test configurations which is generated by P-pre-equivalence relation and R-equivalence relation (resp. P-pre-equivalence relation, Q-pre-equivalence relation and R-equivalence relation) and call it T-equivalence relation (resp. U-equivalence relation).

A T-equivalent class of test configuration  $(\mathcal{X}, \mathcal{L})$  will be called a *test degeneration* and we write it as  $\overline{(\mathcal{X}, \mathcal{L})}$ . Note that, by the base change, central fiber does not change scheme-theoretically (including polarization), which is the origin of this name. Note that, exponents are well-defined for each test degeneration.

A U-equivalent class of test configuration will be called a *test class*. Of course, a test degeneration gives a test class.

Let us assume the polarization  $L$  attached to  $X$  is *very ample*. Then, we have the following. We explain the rigorous meaning in subsections 3.1, 3.2.

**Theorem 2.3.** *Test degenerations  $\overline{(\mathcal{X}, \mathcal{L})}$  with very ample  $\mathcal{L}$  of exponent 1 are parametrized by the set of rational points  $|\Delta(\mathrm{GL}(H^0(X, L)))|_{\mathbb{Q}}$  of the spherical building  $\Delta(\mathrm{GL}(H^0(X, L)))$ .*

Here, the spherical building  $\Delta(\mathrm{GL}(H^0(X, L)))$  is a certain simplicial complex. Its geometric realization  $|\Delta(\mathrm{GL}(H^0(X, L)))|$ , which we denote as  $\Delta_r$ , with a natural topology is known to have a certain natural compatible metric structure (cf. [AB08, Chapter 12]). We will give the definition of the building in subsection 3.1. We will also see that although it is non-compact with respect to the usual topology, *it will be compact* with certain “much stuck” topology to be introduced later. Without fixing exponents, we thus also have:

**Corollary 2.4.** *Test classes of  $(X, L)$  are parametrized by  $\Delta_{\infty} := \varinjlim |\Delta(\mathrm{GL}(H^0(X, L^{\otimes r})))|_{\mathbb{Q}}$ .*

Here, the inductive system is given by taking power of polarization of test configurations i.e.,  $(\mathcal{X}, \mathcal{L}) \rightsquigarrow (\mathcal{X}, \mathcal{L}^{\otimes l})$  with  $l \in \mathbb{Z}_{>0}$ . Note that these  $\Delta_r$  and  $\Delta_{\infty}$  can be expected to correspond to space of geodesics of (finite-dimensional or infinite-dimensional) symmetric spaces.

For the basics of stability, we refer to [Mum65], [Od11]. Based on the continuous (and piecewise-linear) behavior of GIT weights [Mum65, Prop 2.14], we see the continuity of normalized Chow weights:

**Proposition 2.5.** *The normalized Chow weights form continuous function on  $|\Delta(\mathrm{GL}(H^0(X, L^{\otimes r})))|_{\mathbb{Q}}$  with respect to the usual topology. On the other hand, it is a lower semi-continuous function with regards to the “stuck” topology.*

For the definitions of stuck topology and normalized Chow weights, see subsection 3.1 and section 4 respectively. We also have the following abstract existence theorems of “optimal” destabilizing objects, as direct consequences of (the proof of) Theorem 2.3 combined with [Kem78, Theorem 3.4] (cf. also [Rou78]).

**Corollary 2.6** (of Theorem 2.3). *If  $[X \subset \mathbb{P}]$  is Chow-unstable, it has a unique maximally destabilizing test degeneration.*

**Corollary 2.7.** *If  $(X, L)$  is K-unstable, it has a unique maximally destabilizing series of test degenerations.*

For the definitions of these “maximally destabilizing” objects, we will explain in section 4 but they are essentially with regard to the restricted action of  $\mathrm{SL}(H^0(X, L^{\otimes r}))$ . Note that, since if  $(X, L)$  is K-unstable,  $X \hookrightarrow \mathbb{P}(H^0(X, L^{\otimes r}))$ <sup>1</sup> is Chow unstable for sufficiently large  $r \in \mathbb{Z}_{>0}$  (cf. e.g. [RT07, section 3]), Corollary 2.7 simply follows from Corollary 2.6. It would be interesting to know the

<sup>1</sup>In the present notes and other former papers of the author, the notation for the projectivization  $\mathbb{P}(-)$  of vector space is the contravariant one, following Grothendieck.

behavior of Donaldson-Futaki invariants on  $\Delta_\infty$  or to compare Corollary 2.7 with the frameworks of newer stability notions of [Don10] and [Sze11].

It would be also interesting to discuss relation with the similar existence statement in [Sze08, Theorem4] which was for toric case. He used a certain compactness property of the space of convex functions on the Delzant polytope (for which he refers to [Don02, Proposition 5.2.6]).

### 3. PARAMETER SPACE

**3.1. Review of spherical building.** See [Tit74], [AB08] for more detailed accounts for the materials presented in this subsection. In this subsection,  $G$  is assumed to be an arbitrary reductive algebraic group.

**Definition 3.1.** The *spherical building*  $\Delta(G)$  of  $G$  is an abstract simplicial complex whose vertices represent maximal (proper) parabolic subgroups and faces represent finite set of maximal parabolic subgroups whose intersection is again parabolic.

Then, it is known that the following holds.

**Proposition 3.2** (cf. [Mum65, Chapter 2, section 2]). *The rational points of spherical building of  $G$  has a natural correspondence with classes of one parameter subgroups as follows. We denote the center subgroup of  $G$  as  $Z$  and “1-ps” stands for one parameter subgroup.*

$$|\Delta(G)|_{\mathbb{Q}} \cong \{1\text{-ps } \lambda: \mathbb{G}_m \rightarrow G \text{ whose image is not in } Z\} / \sim.$$

Here, the equivalence relation is generated by the following three relations:

- (i)  $\lambda^m \sim \lambda^n$  for any  $m, n \in \mathbb{Z}_{>0}$
- (ii)  $\lambda \sim p\lambda p^{-1}$  if  $p \in (P(\lambda))_k$ , where  $P(\lambda)$  is a parabolic subgroup of  $G$  whose  $k$ -valued points are:  
 $\{\gamma \in (G)_k \mid \lambda(t)\gamma\lambda(t^{-1}) \text{ has a specialization in } G \text{ when } t \text{ specializes to } 0\}.$
- (iii)  $\lambda \sim \lambda\xi$  where  $\xi$  is a one parameter subgroup of  $Z$ .

Note that by the relation (iii), and not thinking one parameter subgroup with image in  $Z$ , we are essentially considering semisimple part. Of course, fixing a maximal torus  $T$ , we have a natural map  $|\Delta(T)|_{\mathbb{Q}} \rightarrow |\Delta(G)|_{\mathbb{Q}}$  which is indeed injective. Thus,  $|\Delta(G)|_{\mathbb{Q}}$  is a union of  $S(T, \mathbb{Q}) := (\text{Hom}_{\text{alg.grp}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{Q}) / \mathbb{Q}_{>0}$  which is dense in  $S(T, \mathbb{R}) := (\text{Hom}_{\text{alg.grp}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}) / \mathbb{R}_{>0}$  (apartments) naturally homeomorphic to sphere.

It is not compact in the usual topology (see [AB08, Chapter 12] for compatible metric structure and its developments) but it is compact if we introduce a new topology as follows.

Note that a variety  $G/N(T)$  parametrizes set of maximal tori where  $N(T)$  denotes the normalizer (algebraic) subgroup of a fixed maximal torus  $T$  of  $G$ . Thus, we have a natural map  $\phi: S(T, \mathbb{R}) \times (G/N(T))_k \rightarrow |\Delta(G)|$ . Using this, we can define our new topology as its associated quotient (compact) topology of the product of the Euclidean topology on  $S(T, \mathbb{R})$  and the Zariski topology on  $(G/N(T))_k$ , via  $\phi$ . We call it *stuck topology* of  $|\Delta(G)|$  (or  $|\Delta(G)|_{\mathbb{Q}}$ , defined by restriction) in this notes. We fix the above notation from now on.

**3.2. Proof of Theorem 2.3.** Note that we are interested in the case  $G = \mathrm{GL}(H^0(X, L))$  which we assume in this subsection.

As Proposition 3.2 shows, the rational points of the spherical building  $|\Delta(G)|_{\mathbb{Q}}$  represent the classes of one parameter subgroups of  $\mathrm{GL}(H^0(X, L))$  with respect to the equivalence relation generated by (i), (ii), (iii) which are nontrivial.

On the other hand, let us recall that each test configuration  $(\mathcal{X}, \mathcal{L})$  with very ample  $\mathcal{L}$  of exponent 1 and projection morphism  $p: \mathcal{X} \rightarrow \mathbb{A}^1$  is isomorphic to the DeConcini-Procesi family of a one parameter subgroup ([RT07, Proposition 3.7]). More precisely, if we fix  $\mathbb{G}_m$ -equivariant trivialization  $f: p_*\mathcal{L} \cong \mathcal{O}^{\oplus(M+1)}$  of  $p_*\mathcal{L}$  then we can take a one parameter subgroup  $\lambda_f$  whose corresponding DeConcini-Procesi family is isomorphic to the original  $(\mathcal{X}, \mathcal{L})$ .

Let us fix a  $\mathbb{G}_m$ -equivariant trivialization  $f$  and take associated eigenspace decomposition of  $H^0(X, L)$  as  $H^0(X, L) = \bigoplus_i V_i$  where the eigenvalue of the action on  $V_i$  is  $t^{w_i}$ . Then, as explained in [Don05, a few lines after Lemma 2], any other choice is given by  $p(t) \circ f$  where  $p \in (P(\lambda))_k$  and  $p(t)$  is defined by  $p(t)|_{\mathrm{End}(V_i, V_j)} = t^{w_j - w_i} p|_{\mathrm{End}(V_i, V_j)}$ . Note that  $\lambda_{p(t) \circ f} = p \lambda_f p^{-1}$ .

Thus, each test configuration is an equivalence class of one parameter subgroups divided by the relation (ii). And it is also easy to see that the relation (i) (resp. (iii)) corresponds to the base change, i.e. P-pre-equivalence (resp. R-equivalence) introduced in Definition 2.2. Hence, test degenerations with very ample polarization of exponent 1 correspond to one parameter subgroups of  $\mathrm{GL}(H^0(X, L))$  modulo (i), (ii), (iii) so that they correspond to rational points of the building  $|\Delta(G)|_{\mathbb{Q}}$ .

**3.3. Almost trivial test configurations.** In [LX11], it is recently pointed out that in the definition of K-stability, we should allow some non-trivial but “almost trivial” test configurations having GIT weights to vanish, or simply dismiss those test configurations of a specific type from our mind. Rigorously speaking, it is characterized by the following condition. Indeed, those should have vanishing Donaldson-Futaki invariants by the formula [Wan08], [Od11].

**Definition 3.3** ([Stp11, Definition 1]). A test configuration  $(\mathcal{X}, \mathcal{L})$  is said to be *almost trivial*<sup>2</sup> if  $\mathcal{X}$  is  $\mathbb{G}_m$ -equivariantly isomorphic to the trivial test configuration, away from a closed subscheme of codimension at least 2.

By taking the fact that they are not necessarily trivial (cf. [LX11, Example 1]) into account, K-stability is slightly reformulated as follows.

**Definition 3.4** (cf. [LX11, subsection 2.2], [Stp11, Definition 2]). A polarized variety  $(X, L)$  is K-stable if  $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$  for all test configurations *which are not almost trivial*.

Now, we give a partial characterization of “almost triviality” from different points of view as follows. We note that the converse does not hold in general.

**Proposition 3.5.** *Assume  $X$  is reduced, equidimensional and embedded into a projective space  $\mathbb{P}(H^0(X, L^{\otimes r}))$  by the complete linear system  $|L^{\otimes r}|$  with some  $r \in \mathbb{Z}_{>0}$ . Take a diagonalization of the  $\mathbb{G}_m$ -action  $\lambda$  on  $H^0(X, L^{\otimes r})$  by the eigenvectors*

<sup>2</sup>It is called *trivial in codimension 2* in [Stp11] but I use this terminology by the following reason. That is, as Stoppa taught me, the way of using phrase as “... in codimension 2” may be familiar to differential geometers but it is not common for algebro-geometers and may well be confusing for them. The meanings are completely the same.

$X_0, \dots, X_c, X_{c+1}, \dots, X_N$  where  $\mathbb{G}_m$  acts on  $X_i$  by  $t^{a_i}$ . Assume that  $a_0 = \dots = a_c < a_{c+1} \leq \dots \leq a_N$ .

Under the assumptions above, if the associated test configuration is almost trivial, then  $c > \dim(X)$  holds and the linear subspace  $V(X_0, \dots, X_c)$  of  $\mathbb{P}(H^0(X, L^{\otimes r}))$ , where  $X_0, \dots, X_c$  vanish, does not intersect with  $X$ , i.e.,  $V(X_0, \dots, X_c) \cap X = \emptyset$ .

*Proof.* First, note that almost test conguration has a property that the linear projection of  $X$  from  $V(X_0, \dots, X_c)$  is birational. Let us assume  $V(X_0, \dots, X_c) \cap X \neq \emptyset$  and prove that the associated test conguration  $\mathcal{X}_\lambda$  is not almost trivial by showing the non-birationality of the linear projection. Assume the contrary and take a closed point  $x$  of  $V(X_0, \dots, X_c) \cap X$ , and denotes the linear projection from  $x$  by  $\pi$ . As the linear projection from  $V(X_0, \dots, X_c)$  is a composite of the linear projection from  $x$  and another linear projection, it is enough to show non-birationality of  $\pi|_X$ .

Note that  $\pi$  naturally extends to a morphism from the blow up of  $X$ , i.e.  $\tilde{\pi}: Bl_x(X) \rightarrow \pi_*(X)$ , where  $\pi_*(X)$  denotes the reduced scheme whose underlying space is the closure of the image of the linear projection  $\pi$  of  $X$ . Since  $x \in X$ ,  $Bl_x(X) \in Bl_x(\mathbb{P}(H^0(X, L^{\otimes r})))$  intersects with the exceptional divisor  $E$  of  $\varphi: Bl_x(\mathbb{P}(H^0(X, L^{\otimes r}))) \rightarrow \mathbb{P}(H^0(X, L^{\otimes r}))$ . Note that if  $\pi|_X$  was birational, it would imply the following equality of intersection numbers:

$$(1) \quad (\mathcal{O}_{\mathbb{P}(H^0(X, L^{\otimes r}))}(1) \cdot \dim(X) \cdot X) = (\mathcal{O}(1) \cdot \dim(X) \cdot \pi_*(X)),$$

where  $\mathcal{O}(1)$  on the right hand side means the tautological ample line bundle on the projective space to which the linear projection  $\pi$  maps. It is because both equal to  $(\mathcal{O}(1)' \cdot \dim(X) \cdot \pi'_*(X))$  where  $\pi'$  is a linear projection from  $x' \in \mathbb{P}(H^0(X, L^{\otimes r}))$  which is sufficiently close to  $x$  so that  $\pi'$  is also birational, and  $\mathcal{O}(1)'$  is the tautological ample line bundle on the projective space to which the linear projection  $\pi'$  maps.

The right hand side of 1 equals to  $((\varphi^*(\mathcal{O}_{\mathbb{P}(H^0(X, L^{\otimes r}))}(1))(-E)) \cdot \dim(X) \cdot Bl_x(X))$  by the projection formula with respect to  $\pi$  and our assumption of birationality of  $\tilde{\pi}|_X$ . On the other hand, it should be smaller than the left hand side by a simple volume comparison, since  $E$  should intersect with  $Bl_x(X)$  in this case. This gives a contradiction. Thus, the resulting test conguration should not have a central ber which is birational to original  $X$ .

Secondly we will prove that  $c \leq \dim(X)$  implies that the corresponding test conguration can not be an almost trivial test conguration as well. Note that if the linear projection from  $V(X_0, \dots, X_c)$  is birational,  $c = \dim(X)$  follows from it by comparing dimensions. If the linear projection was birational to a projective space, this means  $X \in \mathbb{P}(H^0(X, L^{\otimes r}))$  is a degree 1 closed subscheme by a projection formula. However that degree 1 condition implies it is linear subspace which contradicts with non-degeneracy condition of the embedding. Hence the projection from  $V(X_0, \dots, X_c)$  is not birational. This completes the proof.  $\square$

In other words, if a test conguration which corresponds to the one parameter subgroup  $\lambda$  above is almost trivial then its associated ag ideal  $J \subset \mathcal{O}_{X \times \mathbb{A}^1}$  (cf. [Od11]) is  $(t^{a_c})$ . Indeed, recall that  $\mathcal{J}$  was dented as

$$\mathcal{J}\mathcal{O}_{\mathbb{P} \times \mathbb{A}^1}(1) = \sum_{1 \leq j \leq N, a_i \leq j} \mathcal{O}_{\mathbb{P}} X_i t^j.$$

Conversely, if  $\mathcal{J}$  is of the form  $(t^M)$  with some  $M \in \mathbb{Z}_{>0}$ , the natural dominating morphism from  $B := Bl_{\mathcal{J}}(X \times \mathbb{A}^1) (\cong (X \times \mathbb{A}^1))$  to  $\mathcal{X}$  is isomorphism away from



a closed subscheme of  $\mathcal{X}$  with codimension at least 2. Recall that this is a characterization of the almost triviality. Therefore, given a test configuration which is not almost trivial, the associated ag ideal  $J$  is not of the form  $(t^M)$  for some  $M \in \mathbb{Z}_{>0}$  or, if it is of such a form,  $\mathcal{B} \rightarrow \mathcal{X}$  is not isomorphic in codimension 1. For the latter case, we have  $\text{DF}(\mathcal{X}, \mathcal{M}) > 0$  by the arguments of [RT07, section 5] or the formula for the Donaldson-Futaki invariant [Od11].

On the other hand, once a semi test configuration of blow up type  $(\mathcal{B} := \text{Bl}_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$  is given, we can construct a contracted (ample) test configuration  $(\text{Proj} \oplus_{k \geq 0} H^0(X \times \mathbb{A}^1, \mathcal{J}^k p_1^* L^{\otimes rk}), \mathcal{O}(1))$  with obviously the same Donaldson-Futaki invariant.

In sum, we have the following, which is a corrected proposition of [Od11].

**Corollary 3.6** (cf. [Od11, Corollary 3.11]). *Assume  $X$  is Gorenstein in codimension 1. Then, a polarized variety  $(X, L)$  is K-stable (in the sense of Definition 3.4) if and only if for all semi test configurations of the blow up type in [Od11] (i.e.,  $(\mathcal{B} = \text{Bl}_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$  with  $r \in \mathbb{Z}_{>0}$ , flag ideal  $\mathcal{J}$  which is not of the form  $(t^N)$  and  $\mathcal{B}$  Gorenstein in codimension 1), the Donaldson-Futaki invariant is positive.*

Originally, the author claimed the corresponding statement for original K-stability [Don02] which does not work since he ignored the case  $\mathcal{J} = (t^N)$  so that the blow up morphism  $\mathcal{B} \rightarrow X \times \mathbb{A}^1$  is trivial. The proof works for proving the above statement. I apologize for this inaccuracy.

Thus, with this slightly modified K-stability notion and Corollary 3.6, the results of [Od11] the sequels are justified in the naturally corrected form, or partially work without change.

We explain a locus of the spherical building, inside which these pathologies occur.

**Definition 3.7.** Assume  $T$  is a maximal torus of  $G = \text{GL}(H^0(X, L^{\otimes r}))$  and take the corresponding diagonalizing basis  $X_0, \dots, X_N$  of  $H^0(X, L^{\otimes r})$ . In the rational sphere  $S(T, \mathbb{Q}) := (\text{Hom}_{\text{alg.grp}}(\mathbb{G}_m, T) \otimes \mathbb{Q}) / \mathbb{Q}_{>0}$ , we define a subset

$$S'(T, \mathbb{Q}) := \{(a_i) \mid V(\{X_j \mid a_j = \min_i \{a_i\}\}) \cap X = \emptyset\}.$$

Note that  $S'(T, \mathbb{Q})$  has dense complement as for each  $(a_i) \in S'(T, \mathbb{Q})$  we have  $\#\{j \mid a_j = \min_i \{a_i\}\} > 1$ . It is because the corresponding linear subspace should not be of codimension 1 as  $X \subset \mathbb{P}(H^0(X, L^{\otimes r}))$  is non-degenerate.

**Corollary 3.8** (of Proposition 3.5). *In the geometric realization of the spherical building  $\Delta(G)$ , almost trivial test degenerations are parametrized by  $\phi(\cup_T S'(T, \mathbb{Q}))$  whose complement (in  $|\Delta(G)|_{\mathbb{Q}}$ ) is dense.*

#### 4. WEIGHT FUNCTION

We give the definitions of normalized Chow weights and optimal destabilizing test degeneration as follows. See [Mum65, Chapter 1, 2] to review basics of GIT weights. Assume  $G := \text{SL}(H^0(X, L^{\otimes r}))$  in this section and fix a maximal torus  $T$  of  $G$ . With a Weyl group invariant norm  $|\cdot|$  on  $\text{Hom}_{\text{alg.grp}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$ , denote the normalized GIT weight function by  $\nu_T := \frac{\mu_T}{|\cdot|}$ , where  $\mu_T$  is the usual GIT weight function restricted to the set of one parameter subgroups of  $T$ . Recall that  $\nu_T: |\Delta(T)|_{\mathbb{Q}} \rightarrow \mathbb{R}$  can be naturally extended to  $\nu$  on whole  $|\Delta(G)|_{\mathbb{Q}}$  by taking conjugate which sit in  $T$  for each one parameter subgroup. Note that  $\Delta(G) = \Delta(\text{SL}(H^0(X, L^{\otimes r})))$  can be and will be naturally identified with  $\Delta(\text{GL}(H^0(X, L^{\otimes r})))$  by their definitions.



**Definition 4.1.** For a given embedded projective variety  $[X \subset \mathbb{P}]$  and one parameter subgroup  $\lambda: \mathbb{G}_m \rightarrow \mathrm{SL}(H^0(\mathbb{P}, \mathcal{O}(1)))$ , its *normalized Chow weight* means  $\nu(\bar{\lambda}; \mathrm{Chow}(X \subset \mathbb{P}))$ , where  $\mathrm{Chow}(X \subset \mathbb{P})$  means the associated Chow point.

One of the main issue of this notes is the existence of the following object.

**Definition 4.2.** Assume  $[X \subset \mathbb{P}]$  is Chow unstable. Then, its *maximally destabilizing test degeneration* means the test degeneration which corresponds to a point of  $|\Delta(G)|_{\mathbb{Q}}$  with representing one parameter subgroup in  $\mathrm{SL}(H^0(X, L^{\otimes r}))$  which have the minimum (so negative) normalized Chow weight.

Note that Corollary 2.6 which states its existence and uniqueness simply follows from [Rou78], [Kem78] combined with Theorem 2.3 of this notes.

Now, we briefly explain the continuity property of normalized Chow weights (Theorem 2.5).

*Proof of Theorem 2.5.* Recall the following Lemma due to Mumford.

**Lemma 4.3** (cf. [Mum65, Chapter 2, Proposition 2.14]). *With finite linear functions on  $\mathrm{Hom}_{\mathrm{alg.grp}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$  with rational coefficients  $l_1, \dots, l_m$ , and a map  $I: G \rightarrow \{\text{non-empty subsets of } \{1, \dots, m\}\}$ , the weight function  $\mu$  is of the form:*

$$\mu_{gTg^{-1}}(\lambda) = \max\{l_i(\bar{\lambda}) \mid i \in I(g)\}.$$

Here,  $I$  satisfies “lower semicontinuity” in the sense that for any  $J \subset \{1, \dots, m\}$ ,  $\{g \in G \mid J \subset I(g)\}$  is open.

Although the last sentence is not explicitly written in Mumford’s exposition, it is essentially proved there. See also [Kem78]. (He used his own terminology “states” which corresponds to our map  $I$ . ) The continuity of the right hand side implies continuity of normalized Chow weights  $\nu$  on  $|\Delta(G)|_{\mathbb{Q}}$  and moreover, the last sentence implies lower semi-continuity with respect to the stuck topology.  $\square$

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